THE FIRST COHOMOLOGY GROUP $H^1(G, M)$

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Abstract. This paper characterizes the first cohomology group $H^1(G, M)$ where M is a Banach space (with norm $|| \, ||_M$) that is also a left $\mathbb{C}G$ module such that the elements of G act on M as continuous $\mathbb{C}$-linear transformations. We study this group for G an infinite, finitely generated group. Of particular interest are the implications of the vanishing of the group $H^1(G, M)$. The first result is that $H^1(G, \mathbb{C}G)$ imbeds in $H^1(G, M)$ whenever $\mathbb{C}G \subset M \subset L^p(G)$ for some $p \in \mathbb{N}$. This is an unpublished result and shows immediately that if $H^1(G, M) = 0$, then G can have only 1 end. Secondly (also a new result), we show that $H^1(G, M)$ is not Hausdorff if and only if there exist $f_i \in M$ with norm 1 ($||f_i||_M = 1$) for all i with the property that $||gf_i - f_i||_M \longrightarrow 0$ as $i \longrightarrow \infty$ for every $g \in G$. This is then used to show that if M and $|| \, ||_M$ satisfy certain properties and if G satisfies a "strong Følner condition," then $H^1(G, M)$ is not Hausdorff. For the second half of this paper, we give several applications of these last two theorems focusing on the group $G = \mathbb{Z}^n$.

1. Introduction

Motivation for this paper comes from two papers, one by Mohammed E.B. Bekka and Alain Valette ([3]) and the other, an expository paper, by Edward G. Effros ([2]). The first paper examines the group $H^1(G, L^2(G))$ and focuses on the implications of the vanishing of this group. It shows the following:

(1) $H^1(G, L^2(G))$ is Hausdorff if and only if G is non-amenable.

(2) The G-module imbedding $\mathbb{C}G \to L^p(G)$ induces an imbedding of $H^1(G, \mathbb{C}G)$ into $H^1(G, L^p(G)), p \geqslant 1$.

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(3) If $H^1(G, L^2(G)) = 0$, then G is non-amenable with just one end.

The first of these results is due in part to the following result by A. Guichardet ([1],Corollary 2.3 of Chapter III): $H^1(G, L^p(G))$ is not Hausdorff if and only if there exists a sequence e_n in $L^p(G)$ such that $||e_n||_p = 1$ for all n with the property that $||ge_n - e_n||_p \longrightarrow 0$ for all $g \in G$. We show that $L^p(G)$ (with norm $||\ ||_p$) may be replaced by any Banach space M (with norm $||\ ||_M$) that is a $\mathbb{C}G$ module and has the property that the elements of G act on M as continuous $\mathbb{C}$-linear transformations. This is not trivial from Guichardet's theorem. In fact, the topology on $H^1(G, L^p(G))$ (induced by the L^p-norm topology on $L^p(G)$) is entirely different from the topology on $H^1(G, M)$ which is induced by the norm topology on M. This can be used to show that if G satisfies a "strong Følner condition," then $H^1(G, M)$ is not Hausdorff.

As for the second result, we also show that when $\mathbb{C}G \subset M \subset L^p(G)$ for some p $\in \mathbb{N}$, the G-module imbedding $\mathbb{C}G \to M$ induces an imbedding of groups $H^1(G, \mathbb{C}G) \to H^1(G, M)$. This (along with a minor result in [3]) shows that if $H^1(G, M) = 0$, then G can have only one end.

The second paper (by E.G. Effros) characterizes the $\mathbb{C}$ algebra $C^*_{red}(\mathbb{Z}^n)$ and provides a very explicit description of this algebra. He shows first that $C^*_{red}(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$ as $\mathbb{C}$ algebras (where $\mathbb{T}^n$ is the n-torus). Showing that $C^*_{red}(\mathbb{Z}^n)$ has no non-trivial idempotents of course then shows that neither does $C(\mathbb{T}^n)$, which shows that $\mathbb{T}^n$ is connected. Though this result is completely trivial, it ensues some interesting mathematics and gives rise to the paper's title: *Why the Circle is Connected: An Introduction to Quantized Topology.* What Effros's paper does mainly for ours is give us an explicit isomorphism from $C(\mathbb{T}^n)$ onto $C^*_{red}(\mathbb{Z}^n)$.

Our first criterion for the Hausdorffness of $H^1(G, M)$ (Theorem 3) and the isomorphism between $C(\mathbb{T}^n)$ and $C^*_{red}(\mathbb{Z}^n)$ allow us to show that for $n \geqslant 1$, $H^1(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n))$ is not Hausdorff and so $\dim_{\mathbb{C}} H^1(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n)) = \infty$.

Secondly, the isomorphism from $C(\mathbb{T}^n)$ onto $C^*_{red}(\mathbb{Z}^n)$ allows us to describe the groups $H^n(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n))$ explicitly. This in turn shows that there is a natural sequence of imbeddings

$$0 \to H^1(\mathbb{Z}, C^*_{red}(\mathbb{Z})) \to H^2(\mathbb{Z}^2, C^*_{red}(\mathbb{Z}^2)) \to H^3(\mathbb{Z}^3, C^*_{red}(\mathbb{Z}^3)) \to \dots$$

This and the fact that $H^1(\mathbb{Z}, C^*_{red}(\mathbb{Z}))$ is not Hausdorff allows us to show that for each $n \geqslant 1$, $\dim_{\mathbb{C}} H^n(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n)) = \infty$.

Finally, our second criterion for the Hausdorffness of $H^1(G, M)$ (Theorem 4) gives another (more simple) proof that $H^1(\mathbb{Z}, C^*_{red}(\mathbb{Z}))$ is not Hausdorff.

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2. Preliminaries and Definitions

Let G be an infinite, finitely generated group.

Definition 1. *By a G module M, we will mean a $\mathbb{C}$ vector space M along with a homomorphism of G into $Aut(M)$.*

Definition 2. *G satisfies the strong Følner condition means that for every finite subset S of G and every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $M \in \mathbb{N}$ there exists a finite subset X of G such that $|X| > M$ and $|X - g \cdot X| < N$ for every $g \in S$.*

Because this definition may seem tedious, we should note that if G satisfies this condition, then G must be amenable. Also, a simple calculation shows that $\mathbb{Z}$ satisfies the strong Følner condition.

Definition 3. *$\mathbb{C}G$ is the free $\mathbb{C}$ vector space with basis G. In other words, $\mathbb{C}G$ is the set of all finite formal sums of the form $\sum_g a_g g$ where the a_g are in $\mathbb{C}$ and the g are in G.*

$\mathbb{C}G$ is a G module in the obvious way (G acts on an element of $\mathbb{C}G$ by multiplication).

Definition 4. *For $1 \leqslant p < \infty$, $L^p(G)$ is the set of all formal sums (not necessarily finite) of the form $\sum_g a_g g$ with the property that $\sum_g |a_g|^p < \infty$.*

Definition 5. *The norm $|| \, ||_p$ on $L^p(G)$ is given by*

$$|| \sum_g a_g g||_p = [\sum_g |a_g|^p]^{1/p}.$$

$L^p(G)$ is complete in the topology induced by the norm $|| \, ||_p$, so $L^p(G)$ is always a Banach space. There is a multiplication defined on $\mathbb{C}G$ in the natural way ($a_i g_i \cdot a_j g_j = (a_i a_j)(g_i g_j)$). While $\mathbb{C}G$ is a ring (under componentwise addition and this multiplication as its ring multiplication), $L^2(G)$ is not necessarily a ring. One can have $\alpha, \beta \in L^2(G)$ with $\alpha \cdot \beta \notin L^2(G)$. However, it is well known that if $\alpha \in \mathbb{C}G$ and $\beta \in L^2(G)$, then $\alpha \cdot \beta \in L^2(G)$. $L^2(G)$ is certainly a Banach space over $\mathbb{C}$ and via this multiplication, $\mathbb{C}G$ may be considered a subset of $B(L^2(G))$ (the bounded linear operators on $L^2(G)$).

Definition 6. *The operator norm of an element $\alpha \in \mathbb{C}G$ is given by*

$$||\alpha||_{op} = Sup\{||\alpha \cdot \beta||_2 : \beta \in L^2(G), ||\beta||_2 = 1\}.$$

Definition 7. *$C^*_{red}(G)$ is the metric space completion of $\mathbb{C}G$ under the operator norm $|| \, ||_{op}$.*

We immediately have that $C^*_{red}(G)$ is composed entirely of bounded linear operators on $L^2(G)$ and that $C^*_{red}(G)$ is complete and therefore a Banach space. In addition, $\mathbb{C}G \subset C^*_{red}(G) \subset L^2(G)$. We also know that G acts on $C^*_{red}(G)$ (in the obvious way) as continuous $\mathbb{C}$-linear transformations.

Now we turn our attention to the groups $H^1(G, M)$. We will view these groups in two different (though of course equivalent) ways.

First, let A_G be the set of set maps $f\colon G \to M$ with the property that for all a and b in G, $a \cdot f(b) - f(ab) + f(a) = 0$. Let B_G be the set of all such maps given by $f(b) = b \cdot \alpha - \alpha$ for some fixed $\alpha \in M$. Then we have the following first definition of $H^1(G,M)$.

Definition 8. *$H^1(G,M)$ is the quotient group A_G/B_G.*

Now consider the ring $\mathbb{Z}G$ of all finite formal sums of the form $\sum_g z_g g$ with $z_g \in \mathbb{Z}$ for all $g \in G$. Recall that for an arbitrary ring R, an R module is projective if and only if it is a direct summand of a free R module (there are several definitions). We say that an infinite exact sequence of $\mathbb{Z}G$ modules $\cdots \to P_1 \to P_0$ is a projective resolution of $\mathbb{Z}$ as $\mathbb{Z}G$ modules if it extends to an exact sequence $\cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$ of $\mathbb{Z}G$ modules. The $\mathbb{Z}G$ module structure on $\mathbb{Z}$ is given by $g \cdot z_i = z_i$ for $g \in G, z_i \in \mathbb{Z}$. Let $\ldots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{Z} \to 0$ be an exact sequence of $\mathbb{Z}G$ modules with $\cdots \to P_1 \to P_0$ a projective resolution of $\mathbb{Z}$. This first sequence induces another sequence

$$0 \to Hom_{\mathbb{Z}G}(\mathbb{Z}, M) \xrightarrow{d_0^*} Hom_{\mathbb{Z}G}(P_0, M) \xrightarrow{d_1^*} Hom_{\mathbb{Z}G}(P_1, M) \xrightarrow{d_2^*} \ldots$$

which gives us our second definition of $H^1(G,M)$.

Definition 9. $H^1(G,M) = \ker(d_1^*)/\operatorname{im}(d_0^*)$.

As it turns out, this definition is independent of the choice of projective resolution $\ldots \longrightarrow P_1 \longrightarrow P_0$ of $\mathbb{Z}$ as $\mathbb{Z}G$ modules. For the final part of our paper, we will also need the following definition of $H^n(G,M)$.

Definition 10. *For $n \geqslant 1$, $H^n(G,M) = \ker(d_n^*)/\operatorname{im}(d_{n-1}^*)$.*

Note that the two definitions of $H^1(G,M)$ are equivalent.

To define the topology on $H^1(G,M)$, we employ the first definition of this group, A_G/B_G. The topology is induced by the topology of point-wise convergence on A_G. That is for $f_n \in A_G, f_n \longrightarrow 0$ means that $f_n(g) \in M$ converges to 0 in the norm

$|| \, ||_M$ on M for every $g \in G$. We should note that this is where the topologies on $H^1(G, L^2(G))$ and on $H^1(G, M)$ differ. The basic open sets in A_G are the $f \in A_G$ such that $||f(g_1) - a_1||_M, \ldots, ||f(g_n) - a_n||_M < \varepsilon$ for some choice of fixed $g_i \in G, a_i \in M, \varepsilon > 0$ and $n \in \mathbb{N}$. In other words, the set of all such sets forms a basis for the topology on A_G.

Finally, we will need to view $C(\mathbb{T}^n)$ as a subset of $C(\mathbb{T}^{n+1})$ (especially in the proof of Theorem 8). We do this as follows. For $f \in C(\mathbb{T}^n)$ and $(z_1, \ldots, z_{n+1}) \in \mathbb{T}^{n+1}$, define $f(z_1, \ldots, z_{n+1}) = f(z_1, \ldots, z_n)$.

3. The group $H^1(G, M)$

Keep supposing that G is infinite and finitely generated. In addition, suppose that M is a Banach space with norm $|| \, ||_M$ that is a left $\mathbb{C}G$ module and satisfies the property that G acts on M as continuous $\mathbb{C}$-linear transformations. Note that this implies that M is a G module.

Theorem 1. *Suppose that $\mathbb{C}G \subset M \subset L^p(G)$ for some $p \in \mathbb{N}$. Then the G-module imbedding $\mathbb{C}G \to M$ induces an imbedding of groups $H^1(G, \mathbb{C}G) \to H^1(G, M)$.*

Proof: Our result stated above has not been published to date and could prove to be useful. We will, however, follow the proof of Z. Q. Chen as described in [3] (Proposition 1), which shows that the imbedding $\mathbb{C}G \to L^2(G)$ induces an imbedding $H^1(G, \mathbb{C}G) \to H^1(G, L^2(G))$. Since G is finitely generated, suppose that S is a finite generating set for G. For an arbitrary G-module N, we define $C^n(G, N)$ to be the set of all set maps from G^n to N. In the case $n = 0$, we set $C^0(G, N) = N$. We have maps

$$d_0 \colon C^0(G, \mathbb{C}G) \to C^1(G, \mathbb{C}G), \;\; \delta_0 \colon C^0(G, M) \to C^1(G, M) \;\; \text{and}$$

$$d_1 \colon C^1(G, \mathbb{C}G) \to C^2(G, \mathbb{C}G), \delta_1 \colon C^1(G, M) \to C^2(G, M)$$

defined by $[d_0(a)](g) = g \cdot a - a, a \in \mathbb{C}G, [\delta_0(f)](g) = g \cdot f - f, f \in M, [d_1(h)](g_1, g_2) = g_1 h(g_2) - h(g_1 g_2) + h(g_1), [\delta_1(f)](g_1, g_2) = g_1 f(g_2) - f(g_1 g_2) + f(g_1)$. Then

$$H^1(G, \mathbb{C}G) = \frac{\ker(d_1)}{\operatorname{im}(d_0)} \text{ and } H^1(G, M) = \frac{\ker(\delta_1)}{\operatorname{im}(\delta_0)}.$$

The proposed imbedding is of course $a + (im(d_0)) \longmapsto a + (im(\delta_0))$, so define the natural homomorphism

$$\theta\colon \ker(d_1) \to \frac{\ker(\delta_1)}{\operatorname{im}(\delta_0)}$$

via $a \longmapsto a + (\operatorname{im}(\delta_0))$. Since $\mathbb{C}G \subset M$, we have that $\operatorname{im}(d_0) \subset \operatorname{im}(\delta_0)$, so $\operatorname{im}(d_0) \subset \ker(\theta)$.

Let $b \in \ker(\theta)$. Then $\operatorname{im}(b) \subset \mathbb{C}G$ and $b \in \operatorname{im}(\delta_0)$. So there exists $f \in M$ such that for all $g \in G, b(g) = g \cdot f - f \in \mathbb{C}G$. We aim to show that f must lie in $\mathbb{C}G$ and thus $b \in im(d_0)$. Thus, we want to show that f has finite support. Suppose $f = \sum a_g g$. Note that for all $h \in G$, $h \sum a_g g - \sum a_g g = \sum (a_g - a_{hg}) hg \in \mathbb{C}G$. Thus, for for all $h \in G, \varphi(h) = \{g \in G : a_g - a_{hg} \neq 0\}$ is finite. Since S is a finite set, it follows that

$$F(G) = \bigcup_{s \in S} \varphi(s)$$

is finite as well. We may assume that for all $s \in S, s^{-1} \in S$. Let X be the Cayley graph of G with vertex set G and edge set $\{(g, sg) : s \in S\}$. By assuming that S is closed under inverses, it follows that X can be viewed as an undirected graph. Thus, if r and q (elements of G) are connected by an edge, it follows that $r = sq$ for some $s \in S$. Now consider the graph $X - F(G)$. $F(G)$ is finite, so there are finitely many connected components of $X - F(G)$. Then there exists a component of $X - F(G)$ that is infinite (since G is infinite). Let q and r be in this connected component. So there exist $s_1, \ldots, s_n \in S$ such that $s_1 \cdots s_n q = r$, and this path cannot pass through $F(G)$, so we have the property that for all i, $s_i \cdots s_n q \notin \varphi(s)$ for any $s \in S$. Since $s_{i-1} \in S$ for every i, it follows that $a_{s_{i-1} s_i \cdots s_n q} = a_{s_i \cdots s_n q}$ for every i and so $a_q = a_{s_n q} = a_{s_{n-1} s_n q} = \cdots = a_r$. Thus, all of the a_g's are equal for all the g's in $X - F(G)$. By virtue of there being infinitely g's in $X - F(G)$ and since

$f = \sum_g a_g g \in M$ satisfies $\sum_g |a_g|^p < \infty$, it follows that for all g in this connected component of $X - F(G)$, $a_g = 0$. Thus, if $a_t \neq 0$, it follows that t lies in one of the finite connected components of this graph, and there are only finitely many such components (Since $F(G)$ is finite), so there are only finitely many such t. i.e. $a_i = 0$ for all but finitely many i, and thus $f \in \mathbb{C}G$. □

Lemma 1. $\dim_{\mathbb{C}} H^1(G, \mathbb{C}G) = |b(G)| - 1$ *where* $|b(G)|$ *is the number of ends of* G.

Proof: For a proof of this, see [3](Lemma 2). □

Theorem 2. *Suppose that* $\mathbb{C}G \subset M \subset L^p(G)$ *for some* $p \in \mathbb{N}$. *If* $H^1(G, M) = 0$, *then* G *has exactly 1 end.*

Proof: If $H^1(G, M) = 0$, then by Theorem 1, $H^1(G, \mathbb{C}G) = 0$. The previous lemma then tells us that $|b(G)| = 1$ (G has exactly 1 end). □

Theorem 3. $H^1(G, M)$ *is not Hausdorff if and only if there exists* $f_i \in M$ *with norm 1* ($||f_i||_M = 1$) *for all* i *with the property that* $||gf_i - f_i||_M \longrightarrow 0$ *as* $i \longrightarrow \infty$ *for every* $g \in G$.

Proof: For this proof, we think of $H^1(G, M)$ as the set A_G of set maps f from G to M satisfying $a \cdot f(b) - f(ab) + f(a) = 0$ for all $a, b \in G$ modulo the set B_G of maps h of the form $h(g) = g \cdot e - e$ for some $e \in M$. Saying that $H^1(G, M)$ is Hausdorff is equivalent to saying that 0 is closed in $H^1(G, M)$ which is of course equivalent to saying that B_G is closed in A_G. This is equivalent to saying that B_G is complete and is thus a Frechet space. Remember that the topology on A_G is that of point-wise convergence, that is $f_n \colon G \to M$ tends to zero if and only if $f_n(g) \longrightarrow 0$ for every $g \in G$ in the norm $|| \, ||_M$ on M.

We have a continuous one-to-one map $h \colon M \to A_G$ whose image is B_G (the obvious map). Since we are no longer talking about a topology on B_G induced by $|| \, ||_p$, saying that this map is continuous needs justification. Suppose that $(e_n) \in M$ satisfies

$||e_n||_M \longrightarrow 0$. Fix $g \in G$. Then $||[h(e_n)](g)||_M = ||ge_n - e_n||_M \leqslant ||g-1||_M \cdot ||e_n||_M \longrightarrow 0$ as $n \longrightarrow \infty$, since $||g-1||_M$ is fixed. Since the topology on B_G is that of point-wise convergence, it follows that $h(e_n) \longrightarrow 0$, so h is continuous.

Now M is a Banach space, so M is certainly complete and a Frechet space. Since a continuous, bijective map between Frechet spaces has a continuous inverse, saying that B_G is a Frechet space is equivalent to saying that the inverse map from B_G to M is continuous. We claim that this is equivalent to saying that there does not exist a sequence $e_n \in M$ such that such that $||e_n||_M = 1$ for all n and $||ge_n - e_n||_M \longrightarrow 0$ for all $g \in G$.

The inverse map from B_G to M is given by the following. Say $f \in B$. Then f is given by $f(g) = g \cdot e - e$ for some $e \in M$. This inverse map sends f to e. Suppose that there exists such a sequence $e_n \in M$ with $||e_n||_M = 1$ and $||ge_n - e_n||_M \longrightarrow 0$ for all $g \in G$. Thus, the maps in B_G determined by e_n converge to 0 (point-wise). However, their image under this inverse map does not, which means that this map cannot be continuous.

Conversely, suppose that this map is NOT continuous. Noting that the topology on B_G is that of point-wise convergence, this means that there exists a sequence $a_n(g) = ge_n - e_n \in B_G$ such that $||a_n(g)||_M \longrightarrow 0$ for all $g \in G$, but e_n does not converge to 0 in M. Then there exists $\varepsilon > 0$ such that for all $N \geqslant 1$, there exists $m \geqslant N$ such that $||e_m||_M \geqslant \varepsilon$. Multiply e_n by $1/\varepsilon$ and call the new sequence e_n. Note that since $\varepsilon > 0$ is fixed, the sequence of pre-images of the new e_n under this map still converge to 0 for each $g \in G$. Now we can choose a subsequence b_n of e_n such that $||b_n||_M \geqslant 1$ for all n. For each m, multiply b_m by the unique real number $0 < y_m \leqslant 1$ such that the new element of M (rename it X_m) has norm 1. Since $y_m \leqslant 1$ for all m, it follows that for all m, $||X_m||_M = 1$ and $||gX_m - X_m||_M \leqslant ||gb_m - b_m||_M \longrightarrow 0$ for all $g \in G$. $\square$

Theorem 4. *Keep the same assumptions as above about G and M and suppose that $\mathbb{C}G \subset M \subset L^p(G)$ for some $p \geqslant 2$. Suppose further that for every $\alpha \in \mathbb{C}G$, $||\alpha||_1 \geqslant$*

$||\alpha||_M \geqslant ||\alpha||_p$. *If G satisfies the strong Følner condition, then $H^1(G,M)$ is not Hausdorff.*

Proof: Since G is finitely generated, we may let $G = \{g_1, g_2, \dots\}$. For each $n \in \mathbb{N}$, let $G_n = \{g_1, \dots, g_n\}$. Because G satisfies the strong Følner condition, for each $n \in \mathbb{N}$, we may pick $N_n \in \mathbb{N}$ such that for every $M \in \mathbb{N}$, there exists a finite subset X of G such that $|X| > M$ and $|X - g_i \cdot X| < N_n$ for every $g_i \in S_n$. Given any such n, choose a finite subset X_n of G such that $|X_n| > (n \cdot N_n)^p$ and $|X_n - g_i \cdot X_n| < N_n$ for every $g_i \in S_n$. Let

$$\beta_n = \frac{\sum_{x \in X_n} x}{|| \sum_{x \in X_n} x||_M}.$$

Note that for each n, $||\beta_n||_M = 1$. Fix any $g_i \in G$. Then for $n \geqslant i$, we have

$$||g \cdot \beta_n - \beta_n||_M = \frac{||g \sum_{x \in X_n} x - \sum_{x \in X_n} x||_M}{|| \sum_{x \in X_n} x||_M} \leqslant$$

$$\frac{||g \sum_{x \in X_n} x - \sum_{x \in X_n} x||_M}{|| \sum_{x \in X_n} x||_p} \leqslant \frac{||g \sum_{x \in X_n} x - \sum_{x \in X_n} x||_1}{|| \sum_{x \in X_n} x||_p} =$$

$$= \frac{2 \cdot |X_n - g_i \cdot X_n|}{|| \sum_{x \in X_n} x||_p} < \frac{2 \cdot N_n}{|| \sum_{x \in X_n} x||_p} = \frac{2 \cdot N_n}{(|X_n|)^{\frac{1}{p}}} < \frac{2 \cdot N_n}{((n \cdot N_n)^p)^{\frac{1}{p}}} = \frac{2}{n} \to 0.$$

Then $||\beta_n||_M = 1$ for all n and $||g \cdot \beta_n - \beta_n||_M \to 0$ for every $g \in G$. By the previous theorem, $H^1(G,M)$ is not Hausdorff. □

Corollary 1. *Suppose that G satisfies the strong Følner condition. Then $H^1(G, C^*_{red}(G))$ is not Hausdorff and* $\dim_{\mathbb{C}}(H^1(G,M)) = \infty$. □

4. An Application: What Do the Groups $H^1(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n))$ and $H^n(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n))$ look like to $\mathbb{C}$?

Let $S(\mathbb{T}^n)$ be the square integrable functions on $\mathbb{T}^n$. In other words, $S(\mathbb{T}^n)$ is the set of functions $f\colon T^n \to \mathbb{C}$ such that the following integral exists and is finite:

$$\int_{|z_1|=1} \cdots \int_{|z_n|=1} |f(z_1, \dots, z_n)|^2 dz_n \cdots dz_1.$$

Effros's paper ([2]) shows us that we have isomorphisms $\theta_n\colon S(\mathbb{T}^n) \to L^2(\mathbb{Z}^n)$ and $\phi_n\colon C(\mathbb{T}^n) \to C^*_{red}(\mathbb{Z}^n)$ both given by

$$f \longmapsto \sum_{m_1,\ldots,m_n} a_{m_1,\ldots,m_n} Z_1^{m_1} \cdots Z_n^{m_n} \quad \text{where}$$

$$a_{m_1,\ldots,m_n} = \int_{|z_1|=1} \cdots \int_{|z_n|=1} Z_1^{-1-m_1} \cdots Z_n^{-1-m_n} \cdot f(z_1,\ldots,z_n) \cdot \frac{1}{(2\pi i)^n} \cdot dz_n \cdots dz_1$$

For $f \in S(\mathbb{T}^n)$, Effros's paper ([2]) shows us that we also have the following.

$$||\theta_n(f)||_2 = [\frac{1}{(2\pi)^n} \int_{|z_1|=1} \cdots \int_{|z_n|=1} |f(z_1,\ldots,z_n)|^2 dz_n \cdots dz_1]^{1/2}.$$

Lemma 2. *For $f \in C(\mathbb{T}^n)$, $||\phi_n(f)||_{op} = \sup_{z\in\mathbb{T}^n} |f(z)|$.*

Proof: Let d be the usual distance metric on $\mathbb{C}^n$. First, suppose that $g \in S(\mathbb{T}^n)$ satisfies $||\theta_n(g)||_2 = 1$.

$$\Longrightarrow ||\phi_n(f)\theta_n(g)||_2 =$$

$$[\frac{1}{(2\pi)^n} \int_{|z_1|=1} \cdots \int_{|z_n|=1} |g(z_1,\ldots,z_n)|^2 |f(z_1,\ldots,z_n)|^2 dz_n \cdots dz_1]^{1/2} =$$

$$\leqslant \sup_{z\in\mathbb{T}^n} |f(z)| \cdot [\frac{1}{(2\pi)^n} \int_{|z_1|=1} \cdots \int_{|z_n|=1} |g(z_1,\ldots,z_n)|^2 dz_n \cdots dz_1]^{1/2} =$$

$$= \sup_{z\in\mathbb{T}^n} |f(z)|.$$

$$\Longrightarrow ||\phi_n(f)||_{op} \leqslant \sup_{z\in\mathbb{T}^n} |f(z)|.$$

Now, without loss of generality, $||\phi_n(f)||_{op} = 1$. Suppose that $\sup_{z\in\mathbb{T}^n} |f(z)| > 1$. So there is an $\varepsilon > 0$ such that $\sup_{z\in\mathbb{T}^n} |f(z)| > 1 + \varepsilon$. Then there is a $w \in \mathbb{T}^n$ and $1 > \delta > 0$ such that $|f(z)| > 1 + \varepsilon$ for $d(z,w) \leqslant \delta$ where d is the usual distance metric on $\mathbb{C}^n$ (This is due to the continuity of $f(z)$). By $\int \cdots \int_{d(z,w)\leqslant\delta}$, we will mean $\int \cdots \int_{\{d(z,w)\leqslant\delta\}\cap\mathbb{T}^n}$, the integral over the intersection of the δ disk around w (in $\mathbb{C}^n$) with $\mathbb{T}^n$. Since

$$[\frac{1}{(2\pi)^n} \int_{|z_1|=1} \cdots \int_{|z_n|=1} \cdot 1 \cdot dz_n \cdots dz_1]^{1/2} = 1,$$

and since $\delta < 1$, $\{z \in \mathbb{T}^n : d(z,w) \leqslant \delta\}$ is properly contained in $\mathbb{T}^n$. Thus, there exists a $k \in \mathbb{R}^+$ such that $k > 1$ and a $g \in S(\mathbb{T}^n)$ defined by $g(z) = 0$ for $d(z,w) > \delta$ and $g(z) = k$ for $d(z,w) \leqslant \delta$ with the property that

$$||\theta(g)||_2^2 = \frac{1}{(2\pi)^n}\int_{|z_1|=1}\cdots\int_{|z_n|=1}|g(z_1,\ldots,z_n)|^2 dz_n\cdots dz_1 = 1.$$

Also, by the definition of g,

$$\int\cdots\int_{d(z,w)\leqslant\delta}|g(z_1,\ldots,z_n)|^2 dz_n\cdots dz_1 =$$

$$= \int_{|z_1|=1}\cdots\int_{|z_n|=1}|g(z_1,\ldots,z_n)|^2 dz_n\cdots dz_1.$$

Thus,

$$1 = ||\phi_n(f)||_{op} \geqslant ||\theta_n(g)\phi_n(f)||_2 =$$

$$= [\frac{1}{(2\pi)^n}\int_{|z_1|=1}\cdots\int_{|z_n|=1}|g|^2|f|^2 dz_n\cdots dz_1]^{1/2} =$$

$$= [\frac{1}{(2\pi)^n}\int\cdots\int_{d(z,w)\leqslant\delta}|f|^2|g|^2 dz_n\cdots dz_1]^{1/2} =$$

$$\geqslant [\frac{1}{(2\pi)^n}\int\cdots\int_{d(z,w)\leqslant\delta}(1+\varepsilon)^2|g|^2 dz_n\cdots dz_1]^{1/2} =$$

$$= (1+\varepsilon)[\frac{1}{(2\pi)^n}\int\cdots\int_{d(z,w)\leqslant\delta}|g|^2 dz_n\cdots dz_1]^{1/2} =$$

$$= (1+\varepsilon)[\frac{1}{(2\pi)^n}\int_{|z_1|=1}\cdots\int_{|z_n|=1}|g|^2 dz_n\cdots dz_1]^{1/2} = 1+\varepsilon > 1,$$

A contradiction. Thus, $||\phi_n(f)||_{op} = \sup_{z\in\mathbb{T}^n}|f(z)|$. $\square$

Theorem 5. *Let $n \geqslant 1$. Then $H^1(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n))$ is not Hausdorff and so* $\dim_{\mathbb{C}} H^1(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n)) = \infty$.

Proof: Again, let d be the usual distance metric on $\mathbb{C}^n$. Suppose that $H^1(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n))$ is Hausdorff. By Theorem 3, there cannot exist $(f_i) \in C^*_{red}(\mathbb{Z}^n) = C(\mathbb{T}^n)$ such that $||\phi_n(f_i)||_{op} = 1$ for all i with the property that $||\phi_n(gf_i - f_i)||_{op} \longrightarrow 0$ as $i \longrightarrow \infty$ for every $g \in \mathbb{Z}^n$. We will exhibit such an (f_i).

Since given an i, $\{(z_1, \ldots, z_n) \in \mathbb{T}^n : d((z_1, \ldots, z_n), (1, \ldots, 1)) > 1/i\}$ and $\{(z_1, \ldots, z_n) : d((z_1, \ldots, z_n), (1, \ldots, 1)) < 1/(2i)$ for all j$\}$ are disjoint open subsets of $\mathbb{T}^n$ with positive distance between them (this distance is $1/(2i)$), for each i we have a function $f_i \in C(\mathbb{T}^n)$ with image contained in the closed unit disk that is identically 0 on the first set and identically 1 on the second. By the previous lemma, $||\phi_n(f_i)||_{op} = \sup_{z \in \mathbb{T}^n} |f_i(z)| = 1$ for all i. Choose $(m_1, \ldots, m_n) \in \mathbb{Z}^n$. Let $B_i = \{(z_1, \ldots, z_n) \in \mathbb{T}^n : d((z_1, \ldots, z_n), (1, \ldots, 1)) \leqslant 1/i\}$ Then since each f_i is zero on the set $A_i = \{(z_1, \ldots, z_n) \in \mathbb{T}^n : d((z_1, \ldots, z_n), (1, \ldots, 1)) > 1/i\}$,

$$||\phi_n(Z_1^{m_1} \cdots Z_n^{m_n} f_i(z_1, \ldots, z_n) - f_i(z_1, \ldots, z_n))||_{op} =$$

$$= \sup_{(z_1, \ldots, z_n) \in \mathbb{T}^n} |Z_1^{m_1} \cdots Z_n^{m_n} f_i(z_1, \ldots, z_n) - f_i(z_1, \ldots, z_n)| =$$
$$= \sup_{B_i} |Z_1^{m_1} \cdots Z_n^{m_n} f_i(z_1, \ldots, z_n) - f_i(z_1, \ldots, z_n)| \leqslant$$
$$\leqslant \sup_{B_i} |Z_1^{m_1} \cdots Z_n^{m_n} - 1| \cdot Sup_{B_i} |f_i| =$$
$$= \sup_{B_i} |Z_1^{m_1} \cdots Z_n^{m_n} - 1| \longrightarrow 0 \text{ as } \quad i \longrightarrow \infty.$$

This is due to the fact that on $B_i, d((Z_1, \ldots, Z_n), (1, \ldots, 1)) \leqslant 1/i$ and the fact that the $m_1, ..., m_n$ are fixed. In other words, $||gf_i - f_i||_{op} \longrightarrow 0$ as $i \longrightarrow \infty$ for every $g \in \mathbb{Z}^n$. □

Now we turn our attention to the groups $H^n(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n))$. We view $\mathbb{T}^n$ as $\{(z_1, \ldots, z_n) : |z_i| = 1$ for all $i\}$. Again from Effros's paper ([2]), we know that for all n, $C^*_{red}(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$ as $\mathbb{Z}^n$ modules. But what is the $\mathbb{Z}^n$ module structure on $C(\mathbb{T}^n)$? We view $\mathbb{Z}^n$ as the free abelian group on the generators $x_1, \ldots, x_n$. Then the action of an element $x_1^{e_1} \cdots x_n^{e_n}$ on an element $f(z_1, \ldots, z_n)$ of $C(\mathbb{T}^n)$ is given trivially by $(x_1^{e_1} \cdots x_n^{e_n}) \cdot f(z_1, \ldots, z_n) = (z_1^{e_1} \cdots z_n^{e_n}) f(z_1, \ldots, z_n)$. First, we must recall a case of the Kunneth theorem for projective resolutions.

Theorem 6. *(*Künneth Theorem*) Let G be a group and let $P\colon \cdots \to P_1 \xrightarrow{d_1} P_0, Q\colon \cdots \to Q_1 \xrightarrow{\delta_1} Q_0$ be projective resolutions of $\mathbb{Z}$ as $\mathbb{Z}G$ modules. Then $P \otimes Q$ is a projective resolution of $\mathbb{Z}$ as $\mathbb{Z}G$ modules where $[P \otimes Q]_n = \oplus_{i+j=n} P_i \otimes Q_j$. The*

boundary maps ∂_n *are given by* $\partial_n(p_i \otimes q_j) = d_i(p_i) \otimes q_j + (-1)^j p_i \otimes \delta_j(q_j)$ *on simple tensors* $p_i \otimes q_j$.

Proof: This is Theorem V.2.1 of [4]. □

We aim to give an explicit description of the groups $H^n(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n))$ and to show that there is a natural sequence of imbeddings

$$0 \to H^1(\mathbb{Z}, C^*_{red}(\mathbb{Z})) \to H^2(\mathbb{Z}^2, C^*_{red}(\mathbb{Z}^2)) \to H^3(\mathbb{Z}^3, C^*_{red}(\mathbb{Z}^3)) \to \dots$$

In order to do so, we must prove the following lemma.

Lemma 3. *Let* $G = \mathbb{Z}^n$ *be the free abelian group on* $X_1, \dots, X_n$. *Then there exists a projective resolution of* $\mathbb{Z}$ *as* $\mathbb{Z}G$ *modules with exactly n+1 nonzero modules (including* $\mathbb{Z}$*):* $0 \to \mathbb{Z}G \xrightarrow{\rho_n} (\mathbb{Z}G)^n \to \cdots \to \mathbb{Z} \to 0$ *where the map* ρ_n *is given by* $\rho_n(1) = [(X_1 - 1), -(X_2 - 1), \dots, -(X_n - 1)]$.

Proof: Note that for all n, $\mathbb{Z}G = \mathbb{Z}[X_1, \dots, X_n, X_1^{-1}, \dots, X_n^{-1}]$. For $n = 1$, we know that there is a projective resolution of $\mathbb{Z}$: $0 \to \mathbb{Z}G \xrightarrow{d} \mathbb{Z}G \to \mathbb{Z} \to 0$, where d is given by $d(f(X_1)) = (X_1 - 1)f(X_1)$. Suppose that n is at least 2.

Next we must note that for any n, $\mathbb{Z}G \cong \mathbb{Z}\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{Z} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{Z}$ where there are n terms, and the isomorphism is an isomorphism of $\mathbb{Z}G$ modules. The isomorphism $\theta \colon \mathbb{Z}G \to \mathbb{Z}\mathbb{Z} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{Z}$ is given explicitly by $1 \longmapsto 1 \otimes \cdots \otimes 1$. The action of G (the free abelian group on $X_1, \dots, X_n$) on $\mathbb{Z}\mathbb{Z} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{Z}$ is as follows:

$$X_1^{r_1} \cdots X_n^{r_n} * a_1 \otimes \cdots \otimes a_n = X_1^{r_1} a_1 \otimes \cdots \otimes X_n^{r_n} a_n.$$

Let us look again at the projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$ module for $n = 1$ given by $0 \to \mathbb{Z}G \xrightarrow{d} \mathbb{Z}G \xrightarrow{aug} \mathbb{Z} \to 0$. In order to properly view this resolution as a resolution, we will call the first $\mathbb{Z}G$ P_1 and the second P_0. Thus, we have a resolution $0 \to P_1 \xrightarrow{d} P_0 \xrightarrow{aug} \mathbb{Z} \to 0$, where $P_1 = P_0 = \mathbb{Z}\mathbb{Z}$.

The Künneth theorem tells us that the tensor product of resolutions of $\mathbb{Z}$ is again a resolution, so by induction on n, for all n at least 2, there is a resolution of $\mathbb{Z}$:

$$0 \to (P_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} P_1) \xrightarrow{d_n} (P_1 \otimes_{\mathbb{Z}} P_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} P_0) \oplus \cdots \oplus (P_0 \otimes_{\mathbb{Z}} P_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} P_1) \to$$

$$\cdots \to \mathbb{Z} \to 0$$

where everywhere $\ldots$ appears implies that there are exactly n objects being either $\oplus$ed or $\otimes$ed together, except the last $\ldots$ which implies that there are exactly $n+1$ (including $\mathbb{Z}$) nonzero terms in the sequence. The map d_n is given explicitly from the definition of tensor products of resolutions by

$$d_n(a_1 \otimes \cdots \otimes a_n) =$$

$$[(X_1-1)a_n \otimes a_2 \otimes \cdots \otimes a_n, -(a_1 \otimes (X_2-1)a_2 \otimes \cdots \otimes a_n), \ldots, -(a_1 \otimes \cdots \otimes a_{n-1} \otimes (X_n-1)a_n)].$$

Note that the map d_n is just a generalization of our original map $d : P_1 \to P_0$. We know that

$$(P_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} P_1 \otimes_{\mathbb{Z}} P_0) \oplus \cdots \oplus (P_0 \otimes_{\mathbb{Z}} P_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} P_1) \cong (\mathbb{Z}G)^n$$

via a map ζ where the isomorphism is as $\mathbb{Z}G$ modules (and where G is still the free abelian group on $X_1, \ldots, X_n$). The map ζ sends $d_n(1 \otimes \cdots \otimes 1) = [((X_1 - 1) \otimes 1 \otimes \cdots \otimes 1), -(1 \otimes (X_2 - 1) \otimes \cdots \otimes 1), -\ldots, -(1 \otimes \cdots \otimes 1 \otimes (X_n - 1))]$ to $[X_1-1, -(X_2-1), \ldots, -(X_n-1)]$. Translating back into $\mathbb{Z}G$ lingo, it follows directly from the Künneth Theorem that we have a projective resolution of $\mathbb{Z}$ as $\mathbb{Z}G$ modules with exactly $n+1$ (including $\mathbb{Z}$) nonzero modules

$$0 \to \mathbb{Z}G \xrightarrow{\rho_n} (\mathbb{Z}G)^n \to \cdots \to \mathbb{Z} \to 0.$$

The map $\rho_n : \mathbb{Z}G \to (\mathbb{Z}G)^n$ is then of course given by a composition of maps:

$$\rho_n(1) = \zeta(d_n(\theta(1))) = \zeta(d_n(1 \otimes \cdots \otimes 1)) = [X_1 - 1, -(X_2 - 1), \ldots, -(X_n - 1)]. \quad \square$$

Now we are able to give an explicit description of the groups $H^n(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n))$.

Theorem 7. *Let $G = \mathbb{Z}^n$ be the free abelian group on $X_1, \ldots, X_n$. Then we have that*

$$H^n(G, C^*_{red}(G)) \cong \frac{C(\mathbb{T}^n)}{(Z_1 - 1)C(\mathbb{T}^n) + \cdots + (Z_n - 1)C(\mathbb{T}^n)}.$$

Proof: Since for all n, $C^*_{red}(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$, it suffices to prove the statement for $H^n(G, C(\mathbb{T}^n))$ in the place of $H^n(G, C^*_{red}(G))$. From the previous lemma, we have a projective resolution of $\mathbb{Z}$ with $n+1$ nonzero terms (including $\mathbb{Z}$):

$$0 \to \mathbb{Z}G \xrightarrow{\rho_n} (\mathbb{Z}G)^n \to \cdots \to \mathbb{Z} \to 0.$$

From this, we get a new sequence B:

$$0 \to Hom_{\mathbb{Z}G}(\mathbb{Z}, C(\mathbb{T}^n)) \to \cdots \to Hom_{\mathbb{Z}G}((\mathbb{Z}G)^n, C(\mathbb{T}^n)) \xrightarrow{\rho_n^*} Hom_{\mathbb{Z}G}(\mathbb{Z}G, C(\mathbb{T}^n)) \to 0.$$

Using this sequence, we may compute the cohomology group

$$H^n(G, C^*_{red}(G)) = H^n(G, C(\mathbb{T}^n)) = \frac{Hom_{\mathbb{Z}G}(\mathbb{Z}G, C(\mathbb{T}^n))}{\rho_n^*(Hom_{\mathbb{Z}G}((\mathbb{Z}G)^n, C(\mathbb{T}^n)))}.$$

Define $\theta\colon Hom_{\mathbb{Z}G}(\mathbb{Z}G, C(\mathbb{T}^n)) \to C(\mathbb{T}^n)/[(Z_1 - 1)C(\mathbb{T}^n) + \cdots + (Z_n - 1)C(\mathbb{T}^n)]$ by $\phi \longmapsto \phi(1) + [(Z_1 - 1)C(\mathbb{T}^n) + \cdots + (Z_n - 1)C(\mathbb{T}^n)]$. This map is certainly onto, because if $F \in C(\mathbb{T}^n)$, we may just define $\sigma \in Hom_{\mathbb{Z}G}(\mathbb{Z}G, C(\mathbb{T}^n))$ by $\sigma(1) = F$. Now we can compute $ker(\theta)$.

Suppose that $\phi \in Hom_{\mathbb{Z}G}((\mathbb{Z}G)^n, C(\mathbb{T}^n))$. Then

$$\phi(\rho_n(1)) = \phi(X_1 - 1, -(X_2 - 1), \ldots, -(X_n - 1)) =$$

$$(X_1 - 1) * \phi(1, 0, \ldots, 0) + \cdots + (X_n - 1) * \phi(0, \ldots, 0, -1) \in$$

$$\in [(Z_1 - 1)C(\mathbb{T}^n) + \cdots + (Z_n - 1)C(\mathbb{T}^n)].$$

$$\Longrightarrow \rho_n^*(Hom_{\mathbb{Z}G}((\mathbb{Z}G)^n, C(\mathbb{T}^n))) \subset ker(\theta).$$

Now suppose that $\phi \in ker(\theta)$. So $\phi(1) = (Z_1 - 1)f_1 + \cdots + (Z_n - 1)f_n$ for some $f_i \in C(\mathbb{T}^n)$. Define $\pi \in Hom_{\mathbb{Z}G}((\mathbb{Z}G)^n, C(\mathbb{T}^n))$ by $\pi(1, 0, \ldots, 0) = f_1, \pi(0, 1, 0, \ldots, 0) =$

$-f_2, \dots, \pi(0,\dots,0,1) = -f_n$. Then $\pi(\rho_n(1)) = \phi(1)$, and therefore we have that $\phi = \pi(\rho_n) \in \rho_n^*(Hom_{\mathbb{Z}G}((\mathbb{Z}G)^n, C(\mathbb{T}^n)))$.

$$\Longrightarrow H^n(G, C^*_{red}(G)) = H^n(G, C(\mathbb{T}^n)) =$$

$$\frac{Hom_{\mathbb{Z}G}(\mathbb{Z}G, C(\mathbb{T}^n))}{\rho_n^*(Hom_{\mathbb{Z}G}((\mathbb{Z}G)^n, C(\mathbb{T}^n)))} \cong \frac{C(\mathbb{T}^n)}{(Z_1-1)C(\mathbb{T}^n) + \cdots + (Z_n-1)C(\mathbb{T}^n)}. \quad \square$$

Theorem 8. *There is a sequence of group (and $\mathbb{C}$ algebra) imbeddings*

$$0 \to H^1(\mathbb{Z}, C^*_{red}(\mathbb{Z})) \xrightarrow{d_1} H^2(\mathbb{Z}^2, C^*_{red}(\mathbb{Z}^2)) \xrightarrow{d_2} H^3(\mathbb{Z}^3, C^*_{red}(\mathbb{Z}^3)) \to \dots$$

Proof: By the previous theorem, for all $n \geqslant 1$,

$$H^n(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n)) = \frac{C(\mathbb{T}^n)}{(z_1-1)C(\mathbb{T}^n) + \cdots + (z_n-1)C(\mathbb{T}^n)}.$$

Fix an n and define

$$\phi\colon \frac{C(\mathbb{T}^n)}{(z_1-1)C(\mathbb{T}^n) + \cdots + (z_n-1)C(\mathbb{T}^n)}$$

$$\to \frac{C(\mathbb{T}^{n+1})}{(z_1-1)C(\mathbb{T}^{n+1}) + \cdots + (z_{n+1}-1)C(\mathbb{T}^{n+1})}$$

$$\text{via} \qquad f + [(z_1-1)C(\mathbb{T}^n) + \cdots + (z_n-1)C(\mathbb{T}^n)]$$

$$\longmapsto f + [(z_1-1)C(\mathbb{T}^{n+1}) + \cdots + (z_{n+1}-1)C(\mathbb{T}^{n+1})].$$

Since $(z_1-1)C(\mathbb{T}^n)+\cdots+(z_n-1)C(\mathbb{T}^n) \subset (z_1-1)C(\mathbb{T}^{n+1})+\cdots+(z_{n+1}-1)C(\mathbb{T}^{n+1})$, ϕ is certainly well defined. To see that ϕ is injective, we just let $n=1$ (the other cases are identical). If $f \in C(\mathbb{T})$ and $f(z_1) = (z_1-1)f_1(z_1,z_2) + (z_2-1)f_2(z_1,z_2)$ with $f_1, f_2 \in C(\mathbb{T}^2)$, then since f doesn't vary at all with z_2, it follows that we can set $z_2 = 1$ and then we have that $f(z_1) = (z_1-1)f(z_1,1) \in (z_1-1)C(\mathbb{T})$. Thus, $\ker(\phi) = 0$ and so ϕ is injective. $\square$

Theorem 9. *For $n \geqslant 1$,*

$$\dim_{\mathbb{C}} H^n(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n)) = \infty.$$

Proof: For all n, by the last theorem $H^1(\mathbb{Z}, C^*_{red}(\mathbb{Z}))$ imbeds in $H^n(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n))$. Since $\dim_{\mathbb{C}} H^1(\mathbb{Z}, C^*_{red}(\mathbb{Z})) = \infty$, the result follows. □

Finally, using Theorem 4, we are able give a more simple proof of the following result (one proof is given above in Theorem 5).

Theorem 10. $H^1(\mathbb{Z}, C^*_{red}(\mathbb{Z}))$ *is not Hausdorff and* $\dim_{\mathbb{C}} H^1(\mathbb{Z}, C^*_{red}(\mathbb{Z})) = \infty$.

Proof: We know that for $\alpha \in \mathbb{C}\mathbb{Z}$, $||\alpha||_1 \geqslant ||\alpha|_{op} \geqslant ||\alpha||_2$. Thus, Theorem 4 tells us that $H^1(\mathbb{Z}, C^*_{red}(\mathbb{Z}))$ is not Hausdorff. □

5. Our Results (Summarized)

Let G be an infinite, finitely generated group. Let M be a Banach space (with norm $|| \; ||_M$) that is also a left $\mathbb{C}G$ module such that G acts on M as continuous $\mathbb{C}$-linear transformations. In summary, the following useful results were proved in this paper and do not appear in any papers to date.

(1) Suppose that $\mathbb{C}G \subset M \subset L^p(G)$ for some p $\in \mathbb{N}$. Then the G-module imbedding $\mathbb{C}G \to M$ induces an imbedding of groups $H^1(G, \mathbb{C}G) \to H^1(G, M)$.

(2) If $\mathbb{C}G \subset M \subset L^p(G)$ for some p $\in \mathbb{N}$ and if $H^1(G, M) = 0$, then G has exactly 1 end.

(3) $H^1(G, M)$ is not Hausdorff if and only if there exists $f_i \in M$ with norm 1 ($||f_i||_M = 1$) for all i with the property that $||gf_i - f_i||_M \longrightarrow 0$ as $i \longrightarrow \infty$ for every $g \in G$.

(4) If $\mathbb{C}G \subset M \subset L^p(G)$ for some $p \geqslant 2$, $||\alpha||_1 \geqslant ||\alpha||_M \geqslant ||\alpha||_p$ for every $\alpha \in \mathbb{C}G$, and if G satisfies the strong Følner condition, then $H^1(G, M)$ is not Hausdorff and therefore $\dim_{\mathbb{C}} H^1(G, M) = \infty$.

We have used these results (with our motivation coming from Effros's paper ([2])) to show the following for $n \geqslant 1$.

(1) $H^1(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n))$ is not Hausdorff and thus $\dim_{\mathbb{C}} H^1(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n)) = \infty$ (with two proofs for $n = 1$).

(2) $\dim_{\mathbb{C}} H^n(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n)) = \infty$.

(3) There is a natural sequence of imbeddings

$$0 \to H^1(\mathbb{Z}, C^*_{red}(\mathbb{Z})) \to H^2(\mathbb{Z}^2, C^*_{red}(\mathbb{Z}^2)) \to H^3(\mathbb{Z}^3, C^*_{red}(\mathbb{Z}^3)) \to \dots$$

(4)

$$H^1(\mathbb{Z}^n, C^*_{red}(\mathbb{Z}^n)) = \frac{C(\mathbb{T}^n)}{(Z_1 - 1)C(\mathbb{T}^n) + \cdots + (Z_n - 1)C(\mathbb{T}^n)}.$$

6. Conjectures and Future Work

Conjecture 1. *Let G be an infinite, finitely generated group. Let M be a Banach space (with norm $|| \; ||_M$) that is also a left $\mathbb{C}G$ module such that G acts on M as continuous $\mathbb{C}$-linear transformations. Then G is amenable if and only if $H^1(G, M)$ is not Hausdorff.*

We should note that the proof given in [3] of this with $L^2(G)$ is specific to that case.

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